BTZ Black Hole Entropy from Ponzano-Regge Gravity

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Abstract

The entropy of the BTZ black hole is computed in the Ponzano-Regge formulation of three-dimensional lattice gravity. It is seen that the correct semi-classical behaviour of entropy is reproduced by states that correspond to all possible triangulations of the Euclidean black hole. The maximum contribution to the entropy comes from states at the horizon.

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Since Bekenstein's proposal that black holes have entropy [1], there have been many attempts to explain this in terms of microstates associated to the horizon of the black hole. The discovery of a black hole solution in $(2+1)-d$ gravity with a negative cosmological constant by Banados, Teitelboim and Zanelli [2], led to several attempts to compute its entropy. $(2+1)-d$ gravity can be written as a Chern-Simons theory [3]. Using the fact that Chern-Simons theory on a manifold with boundary induces a WZW theory on the boundary [6], [5], an expression for entropy has been obtained by considering the Euclidean extension of the BTZ black hole [7], [4], [8], [9].

Another approach [10] has been to use the fact that that $(2+1)-d$ gravity with a negative cosmological constant has an anti-deSitter space (AdS_3) vacuum solution [11]. The BTZ black hole is obtained from AdS_3 by discrete identifications. Then the BTZ black hole entropy is obtained by considering states in a conformal field theory induced on the boundary of the AdS_3.

Yet another formulation of quantum gravity is provided by the Ponzano-Regge lattice gravity. The issue of the states corresponding to black hole entropy is still not resolved [24]. It is hoped that this formulation might lead to a 'picture' of what the relevant states of a black hole are. In this paper, we shall study the BTZ black hole in this formulation. Then the relevant states are all possible triangulations of the black hole manifold, and give an entropy proportional to the 'area.' The maximum contribution to the entropy comes from states at the horizon. The expression for entropy has an arbitrary parameter.Its origin is similar to that of the Immirzi parameter that appears in the calculation of the entropy of the Schwarzschild black hole in $(3+1)-d$ in the framework of loop gravity [13], [14]. The entropy obtained by us is of the familiar Bekenstein-Hawking form for the same value of the arbitrary parameter as that of the Immirzi parameter in the $(3+1)-d$ calculation.

In the three-dimensional lattice gravity of Ponzano and Regge, [12] , the three-manifold M is decomposed into simplices. Each three-simplex is a tetrahedron. To each edge of the tetrahedron is assigned a half-integral number j such that $(\sqrt{j(j+1)})$ is the discretized length of that edge. (For large j, this becomes $(j+\frac{1}{2})$.) These lengths must of course satisfy the triangle inequalities corresponding to the triangular faces of the tetrahedron. When the lengths are large, the Racah-Wigner $6j$ symbol is related to the Regge action for a tetrahedron [12]. The partition function of Ponzano-Regge gravity is constructed for the manifold M out of the various $6j$ symbols associated with the tetrahedra in the simplicial decomposition of M. It has been shown that the space of states of this theory is the same as that of $ISO(3)$ Chern-Simons theory on M [15].

The q-analogue of the Ponzano-Regge model has been studied by Turaev and Viro [16]. The $q-6j$ symbol for large j has been shown to be related to the Regge action for a tetrahedron for the case of gravity with a cosmological constant [17]. The Turaev-Viro q-analogue would therefore describe gravity with a cosmological constant. The deformation parameter q is related to the cosmological constant. It has also been shown that the Turaev-Viro partition function is

the square of the partition function of SU(2) Chern-Simons theory, where the coupling constant k, is related to the deformation parameter q as $q = \exp \frac{2\pi i}{k+2}$ [18]. The exact relation between the states of the Turaev-Viro model and the Chern-Simons theory is obtained from the one-to-one correspondence between a homotopy class of coloured trivalent networks of Wilson lines and a triangulation with colourings on the sides. The Turaev-Viro partition function, written in terms of Chern-Simons theory is related to the Einstein-Hilbert action for Euclidean gravity as follows: The partition function of the q-analogue lattice model given by the square of $SU(2)$ Chern-Simons partition function, may be rewritten in terms of $SL(2, C)$ Chern-Simons theory as

$$Z_k \quad = \quad \int [dA, \ d\bar{A}] \ \exp[\frac{ik}{4\pi} \int (AdA + \frac{2}{3}A^3) \ - \ (\bar{A}d\bar{A} + \frac{2}{3}\bar{A}^3 \ )] \tag{1}$$

where A is an $SL(2, C)$ connection. As has been shown in [19] and [20], for a manifold with boundary, $Z_{SL(2,C)}[A, \bar{A}]$ can be thought of as $|Z_{SU(2)}[A]|^2$. For the manifold with solid torus topology, the $SL(2, C)$ wave functions can be written in a basis of products of two conjugate $SU(2)$ wave functions [20].

To relate this to $3 - d$ gravity, in terms of the triad e and spin connection field ω, we define $A = \omega + \frac{i \, e}{l}$ and $\bar{A} = \omega - \frac{i \, e}{l}$, and denote $I[A] = \frac{k}{4\pi} \int(AdA + \frac{2}{3}A^3)$. Then, the action in (1) can be written as the Einstein-Hilbert action of gravity with negative cosmological constant.

$$i(I[A] \ - \ I[\bar{A}]) \quad = \quad \frac{1}{16\pi G} \int_M \sqrt{g}(R \ + \ \frac{2}{l^2}) \quad = \quad I_{EH} \tag{2}$$

The cosmological constant is $\Lambda = -\frac{1}{l^2}$ and the coupling constant k given by $k = -\frac{l}{4G}$.

In this paper, we shall study the BTZ black hole in the Turaev-Viro formulation. The BTZ (Lorentzian) black hole metric for $(2+1) - d$ spacetime with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$ has a Euclidean continuation [4] which is given by

$$ds^2 \quad = \quad N^2 \ d\tau^2 + N^{-2} \ dr^2 + r^2 \ (d\phi + N^\phi d\tau)^2 \tag{3}$$

with

$$N \quad = \quad (-M + \frac{r^2}{l^2} - \frac{J^2}{4r^2})^{\frac{1}{2}}, \quad N_\phi \quad = \quad -\frac{J}{2r^2} \tag{4}$$

The inner and the outer horizons of the Lorentzian solution get mapped in the Euclidean continuation to ir_- and r_+ respectively, where

$$r_\pm^2 \quad = \quad \frac{Ml^2}{2} \ [1 \ \pm \ (1 + \frac{J^2}{M^2 \ l^2})^{1/2}] \tag{5}$$

As shown by Carlip and Teitelboim [4], the metric (3) can, by a coordinate transformation, be reduced to the metric for the upper half-space model of hyperbolic three-space $\mathbf{H^3}$. The fact that the Schwarzschild angular coordinate ϕ in (3) is periodic results in the Euclidean black

hole being obtained by discrete identifications in $\mathbf{H^3}$. The Euclidean black hole corresponds to the region *outside* the event horizon of the Lorentzian solution. The topology of this space is $R^2 \otimes S^1$. In [4], a fundamental region corresponding to these discrete identifications is taken to represent the black hole. The topology of this fundamental region is that of a solid torus, i.e $D^2 \otimes S^1$. The horizon is a degenerate circle of radius r_+ at the core of the solid torus. r_- represents the amount of twist made before making the discrete identifications to obtain the solid torus.

We shall calculate the entropy associated with the Euclidean black hole. The question to be addressed, therefore, is what the black hole corresponds to in the Ponzano-Regge framework. Each triangulation of a solid torus is a realisation of the black hole topology in the lattice picture. With each such triangulation with specified lengths on the boundary is associated a 'partition function' as defined by Turaev and Viro [16]. The partition function for a manifold without boundary is given by:

$$Z_{TV} = \sum_{\substack{colourings \ j_e \leq \frac{k}{2}}} \prod_{vertices} \frac{1}{\Lambda_q} \prod_{e:\ edges} (-1)^{2j_e} [2j_e + 1]_q \ \times$$

$$\prod_{t:\ tetrahedra} \exp\left(-i\pi \sum_i j_i(t)\right) \left\{ \begin{array}{ccc} j_1(t) & j_2(t) & j_3(t) \\ j_4(t) & j_5(t) & j_6(t) \end{array} \right\}_q \tag{6}$$

Here, vertices, edges and tetrahedra are those associated with the triangulation. The subscript q and square brackets indicate q numbers instead of ordinary numbers, and $q - 6j$ symbols instead of ordinary $6j$ symbols. $\Lambda_q = \frac{-2(k+2)}{(q^{1/2} - q^{1/2})^2}$.

For a manifold with boundary, in the expression in (6), in addition, there is a factor of $\frac{1}{\sqrt{\Lambda_q}}$ per boundary vertex, and $\exp(i\pi j_b)\sqrt{[2j_b + 1]_q}$ per boundary edge with a spin j_b. As mentioned earlier, Z_{TV} is related to $|Z_{SU(2)}|^2$ which is related to Z_{grav}, the partition function of Euclidean gravity. However, as pointed out in [21], the integration measure in the Chern-Simons partition function is $[dA, d\bar{A}]$, whereas for the gravity partition function, it is $[de, d\omega]$. Since $A = \omega + \frac{ie}{l}$, the relation between the two involves $\frac{1}{l}$ factors. It was argued in [21] that for a closed manifold, the factors of $\frac{1}{\Lambda_q}$ appearing in (6) are to do precisely with the difference in the measures. This can also be extended to the case of a manifold with boundary, because the choice of $\frac{1}{\sqrt{\Lambda_q}}$ per boundary vertex in the Turaev-Viro partition function was made so that on fusing two such manifolds to make a closed manifold, one obtained the partition function (6) for a closed manifold. Therefore, the Turaev-Viro partition function could be thought of as equivalent to the square of a Chern-Simons partition function, but would be equal to a gravity partition function only without the Λ_q terms in (6). The partition function for gravity for a

4

manifold with boundary would be given by

$$Z_{grav} = \sum_{colourings\ j_e \leq \frac{k}{2}} \prod_{e:\ internal edges} (-1)^{2j_e} [2j_e + 1]_q \times$$

$$\prod_{b:\ boundary edges} \exp\left(i\pi j_b\right) \sqrt{[2j_b + 1]}\ \times$$

$$\prod_{t:\ tetrahedra} \exp\left(-i\pi \sum_i j_i(t)\right) \left\{ \begin{array}{ccc} j_1(t) & j_2(t) & j_3(t) \\ j_4(t) & j_5(t) & j_6(t) \end{array} \right\}_q \qquad (7)$$

The expression in (7) is a functional of the triangulation and spins on the boundary.

The black hole has the topology of a solid torus with its longitude given by a radius r_+, and with a twist in the solid torus proportional to r_-. The possible states that could be associated with this black hole are the states associated with different triangulations of the black hole manifold, with the restriction that the longitude have a radius r_+. The other circumference can take all possible values. This can also be understood as follows: The total partition function is formally a path integral over both bulk and boundary metrics - in this case, a sum over spin assignments in the bulk and on the boundary such that the longitude has length $2\pi r_+$. Summing over the spins in the bulk yields an expression of the form (7) which is a functional of the spins on the boundary. Summing over the all boundary spins consistent with the circumference being $2\pi r_+$ would give the black hole partition function. We now proceed to estimate the contributions to this sum. We look at all possible triangulations of this solid torus with different spins on the boundary. Remembering that a spin j corresponds to a length $(\sqrt{j(j+1)})$ in the triangulation , we see that for any given triangulation, the spins will be constrained by the lengths of the circumferences of this solid torus.

We can have an arbitrary triangulation of the solid torus in three steps as follows : In Fig.1, the torus is formed out of blocks, each of which has two N-polygonal faces of the type in Fig.2, that are joined to the faces of the next block. Each of these faces can be triangulated, and corresponding triangles on the opposite face can be joined to these triangles, as in Fig.2, resulting in each block being broken up into a certain number of prisms (six in the case drawn here). Then each prism can be triangulated into tetrahedra, as shown in Fig.3.

These three steps yield a a triangulation of the solid torus. For this triangulation to represent a state of the black hole, however, some of the spins corresponding to the longitudinal cycle must be restricted by the fact that the sum of the lengths associated with them is $2\pi r_+$.

In Fig.1, we see that the spins j_1, j_2 etc. corresponding to the longitudinal cycle have to satisfy

$$\gamma\ l_p \sum_{i=no.\ of\ blocks} (\sqrt{j(j+1)}) = 2\pi r_+ \qquad (8)$$

where the unit of length used is $\gamma\ l_p$. l_p is the Planck length in three dimensions and γ is an arbitrary parameter. There is an ambiguity in the unit of length in the Ponzano-Regge
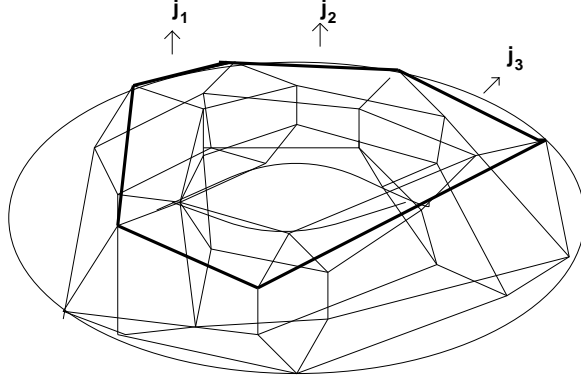
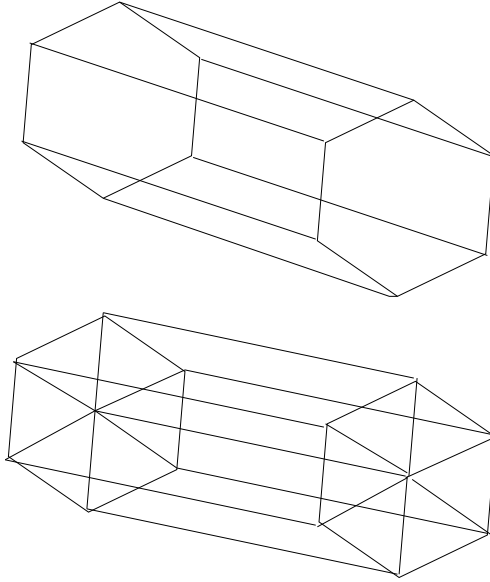Figure 1: *Torus formed out of polygonal blocks*

Figure 2: *Polygonal block broken up into prisms*

formalism itself. The result of a loop gravity calculation by Rovelli [22] suggests that in Ponzano Regge gravity, the unit of length associated with a spin is l_p. In $(3+1) - d$ canonical gravity [26], [27], there is an arbitrary parameter associated with scaling of the canonical variables, and multiplies the expression for area and volume eigenvalues. Here also, we find that there is an arbitrary parameter γ which multiplies the unit of length l_p obtained in [22], and is associated with scaling of e and ω.

We first consider the case of the torus formed out of blocks of prisms, each of which is triangulated as in Fig.3. Then, the spins that are restricted in each prism by constraints of the form (8) are the longitudinal spins j_{AD}, j_{BE} and j_{CF} and their corresponding counterparts in other prisms - the length associated with each of these spins when summed with lengths associated with corresponding spins in all other prisms must give $2\pi r_+$.
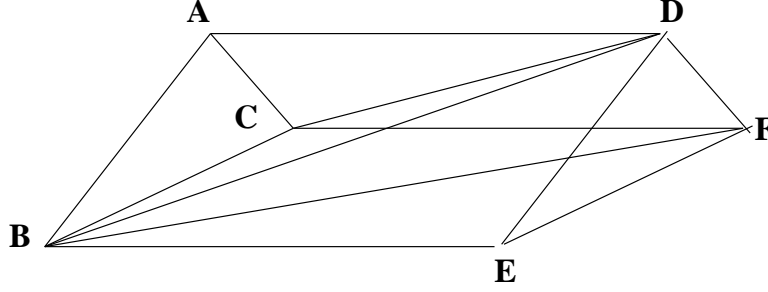
Figure 3: *Triangulation of prism into tetrahedra*

Given this restriction, we try to estimate the partition function contribution from these prisms for all possible values of the various spins. The contribution can be estimated by considering each prism separately, and then multiplying contributions from all prisms. The contribution of each prism with a certain assignment of spins is given from (6) and its modification for the case with a boundary as:

$$
\begin{aligned}
Z_b \quad = \quad & \prod_{u:\ unshared\,edges} \sqrt{[2j_u + 1]}\ \exp\left(i\pi j_u\right) \prod_{s:\ shared\,edges} [2j_s + 1]^{\frac{1}{4}}\ \exp\left(\frac{i\pi j_s}{2}\right) \times \\[6pt]
& \exp\left(-i\pi(j_{AB} + j_{AC} + j_{BC} + j_{CD} + j_{AD} + j_{BD})\right) \left\{ \begin{matrix} j_{AB} & j_{BC} & j_{AC} \\ j_{CD} & j_{AD} & j_{BD} \end{matrix} \right\}_q \times \\[6pt]
& \exp\left(-i\pi(j_{BC} + j_{CF} + j_{BF} + j_{DF} + j_{BD} + j_{CD})\right) \left\{ \begin{matrix} j_{BC} & j_{CF} & j_{BF} \\ j_{DF} & j_{BD} & j_{CD} \end{matrix} \right\}_q \times \\[6pt]
& \exp\left(-i\pi(j_{BE} + j_{EF} + j_{BF} + j_{DF} + j_{BD} + j_{ED})\right) \left\{ \begin{matrix} j_{BE} & j_{EF} & j_{BF} \\ j_{DF} & j_{BD} & j_{ED} \end{matrix} \right\}_q
\end{aligned}
$$

$$(9)$$

Here, j_u refers to the unshared sides and j_s to the sides shared with the blocks to which the prism is fused to form the solid torus.

The expression Z_b corresponds to one prism. For a triangulation with n identical prisms, the contribution would simply be $(Z_b)^n$. It is difficult to calculate the partition function exactly for arbitrary spins associated with the edges. We shall calculate the contribution to the partition function for some specific assignment of spins to argue that dominant contribution comes from a specific assignment below.

The simplest choice of spins we can take is $j_{AB} = j_{AC} = j_{BC} = j_{DE} = j_{EF} = j_{DF} = 0$. Then, the choice of the other spins decides the number of prisms in the triangulation. All the other spins have to be equal, i.e $j_{AD} = j_{CF} = j_{BE} = j_{CD} = j_{BD} = j_{BF} = j$. Geometrically, this corresponds to each prism having collapsed into a line with spin j. Thus, the torus itself collapses to just the longitudinal cycle. Then, from the constraint (8), the number of blocks n

for fixed r_+ is given by

$$n = \frac{2\pi r_+}{\gamma \, l_p \, \sqrt{j(j+1)}} \qquad (10)$$

The largest value of n corresponds to the lowest spin, $j = 1/2$. This in turn yields the maximum contribution to the partition function. Each prism contributes a value $[2]_q$, and therefore, the total contribution to the partition function for large k is 2^n, where $n = \frac{4\pi r_+}{\gamma \, l_p \, \sqrt{3}}$.

Any other assignment of spins yields a contribution less than that of this case. To see this, let us consider the case $j_{AB} = j_{AC} = j_{BC} = j_{DE} = j_{EF} = j_{DF} = R$, $(R \;\; large)$, we can take recourse to the following asymptotic formula for the 6j symbol for $R \gg 1$ [23] valid in our case for large k:

$$\left\{ \begin{array}{ccc} a & b & c \\ R & R & R \end{array} \right\} = \frac{(-1)^c}{\sqrt{2R(2c+1)}} C_{a0b0}^{c0} \qquad (11)$$

where C_{a0b0}^{c0} is the Clebsch-Gordon coefficient.

Here, we find that the largest contribution to the partition function comes again when $j_{AD} = j_{CF} = j_{BE} = 1/2$ and $j_{CD} = j_{BF} = j_{BD} \sim R$. This contribution for large k is $(\sqrt{2})^n$ where n is given by (10), and is smaller than the earlier mentioned contribution.

The contribution from the intermediate values of spins remain to be found. The difficulty with calculating this contribution is due to the fact that the $6j$ symbols in the contribution have to be evaluated. In order to obtain this contribution, we look at the corrections to the asymptotic formula (11) for R not very large. We find that for $R \geq 2$, considering the corrections, we have

$$\left\{ \begin{array}{ccc} a & b & c \\ R & R & R \end{array} \right\} \leq \frac{(-1)^c}{\sqrt{2R(2c+1)}} C_{a0b0}^{c0} \qquad (12)$$

We now take the following choice of spins : $j_{AB} = j_{BC} = j_{DE} = j_{EF} = j_{CD} = R$, $j_{AC} = j_{DF} = j_{BD} = j_{BF} = R - j$, $j_{AD} = j_{CF} = j_{BE} = j$. The contribution of this choice of spins can be estimated using the r.h.s of (12). This can be done in two regimes :

i)$R \gg j$. In this case, $R \geq 2$.

ii) $j \gg R$. Here, $j \geq 2$.

Case ii) is estimated numerically. It is found that case i) has a higher contribution, and its contribution is highest for $j = 1/2$. Further, this contribution is less than $(\sqrt{2})^n$. Case i)

8

and ii) describe those values of spins where some spins are larger than others. There are other choices which can be investigated, e.g those where all the spins have the same value. If this is a large value (≥ 2), again it is seen that this contribution is much lesser than $(\sqrt{2})^n$.

Finally, there remains the case where all spins are small. Here, it is possible to do exact calculations for a large number of cases. In all these cases, it is explicitly found that the contribution is less than $(\sqrt{2})^n$. Some examples are :

a)$j_{AB} = j_{BC} = j_{ED} = j_{EF} = j_{AD} = j_{CD} = j_{CF} = j_{BE} = 1/2,$

$j_{AC} = j_{DF} = j_{BF} = j_{BD} = 1.$

This contributes $(4/5)^n$.

b)$j_{AB} = j_{AC} = j_{DE} = j_{DF} = j_{AD} = j_{CF} = j_{BE} = 1/2,$

$j_{BC} = j_{EF} = j_{BD} = j_{CD} = 1, \quad j_{BF} = 3/2.$

This contributes $(\sqrt{32/27})^n$.

c)$j_{AB} = j_{AC} = j_{BC} = j_{DE} = j_{DF} = j_{EF} = 1,$

$j_{AD} = j_{CF} = j_{BE} = 1/2, \quad j_{CD} = j_{BD} = j_{BF} = 3/2.$

This contributes $(\sqrt{50/27})^n$.

Summing the contributions from the prism triangulation from all these different regimes of spin values, we see that the maximum contribution seems to come from the first case considered, where $j_{AB} = j_{AC} = j_{BC} = j_{DE} = j_{EF} = j_{EF} = j_{DF} = 0$ and $j_{AD} = j_{CF} = j_{BE} = j_{CD} = j_{BD} = j_{BF} = 1/2$, and where the torus collapses into the longitudinal cycle.

We have considered upto now, only the prism triangulation of the torus. As mentioned before, the torus can be triangulated by other polygonal blocks, each of which can be triangulated by breaking the block into prisms and triangulating them. Therefore, many of the simplifying methods used here can also be used to determine the contribution from other polygons. For polygonal blocks with large spins on the polygonal sides, it is possible to estimate the contribution, which is less than $(\sqrt{2})^n$. Also, for some simple polygons (cube, pentagon) with very small values of spins on their polygonal sides, it is possible to explicitly calculate the contribution, again less than $(\sqrt{2})^n$.

The discussion above suggests that the maximum contribution on considering *all* possible triangulations would still come from the term corresponding to the torus collapsing to the longitudinal cycle, i.e *from states at the horizon*. This contribution is $([2]_q)^n$, which for large k is simply 2^n.

We note here that r_- does not appear in the calculation of the partition function contributions. This is because it only represents a twist in the solid torus and does not change the contribution of states associated with a triangulation. The untwisted triangulation that we have considered corresponds to a black hole with angular momentum $J = 0$. However, the same contribution would also come from a twisted triangulation which corresponds to another black hole with $J \neq 0$ that has the same value of r_+. Thus, this contribution is the same for all black holes with horizon radius r_+.

Looking at states at a fixed value of r_+ corresponds to working in the microcanonical ensemble. The entropy is therefore given by the logarithm of the partition function. As mentioned above, the leading contribution to the partition function for large k is 2^n. The entropy is then be mainly due to this term, and $S = n \, ln2$. Since $n = \frac{4\pi r_+}{\gamma \, l_p \, \sqrt{3}}$,

$$S \quad = \quad \frac{2\pi r_+}{\gamma \, 4\pi G \, \sqrt{3}} \, ln2 \tag{13}$$

where $2\pi r_+$ is the length of the horizon.

This expression for entropy has factors similar to that obtained in a different context for the Schwarzschild black hole in $(3+1) - d$ in the framework of loop gravity [13], [14]. As mentioned before, in the loop gravity result, there is an arbitrary parameter in the expression for entropy, which is related to scaling of the canonical variables. This is chosen to have a particular value so that the expression for the entropy matches the Bekenstein-Hawking result. On choosing the *same* value, $\frac{ln2}{\pi \, \sqrt{3}}$, for γ in our expression (13) for the entropy, the entropy assumes the familiar form

$$S \quad = \quad \frac{A}{4G} \tag{14}$$

where A is the 'area' of the horizon, $2\pi r_+$.

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